

# LOCAL ACYCLIC FIBRATIONS AND THE DE RHAM COMPLEX

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ABSTRACT. We reinterpret algebraic de Rham cohomology for a possibly singular complex variety  $X$  as sheaf cohomology in the site of smooth schemes over  $X$  with Voevodsky's  $h$ -topology. Our results extend to the algebraic de Rham complex as well. Our main technique is to extend Čech cohomology of hypercovers to arbitrary local acyclic fibrations of simplicial presheaves.

## 1. INTRODUCTION

Let  $X$  be a separated scheme finite type over the complex numbers  $\mathbb{C}$ . Following Deligne, Du Bois ([4]) constructs the algebraic de Rham complex of  $X$

$$\underline{\Omega}_{X/\mathbb{C}}^\bullet := Re_* \Omega_{X_\bullet/\mathbb{C}}^\bullet$$

by a choice of a smooth proper hypercover. It is well-defined in the filtered derived category. Morally this Čech complex should be a derived direct image from some topos to the Zariski site; showing this is the aim of this paper.

The choice of topos appears to be a delicate matter. Using the topology of “universal cohomological descent” (which we abbreviate “ $ucd$ ”) on proper and smooth schemes turns out to be technically inconvenient. We use instead Voevodsky's  $h$ -topology [21] on possibly open schemes. Denote by  $\mathrm{Sm}_h/X$  the category of smooth separated schemes finite type over  $X$ , equipped with the  $h$ -topology. We show the presheaf  $\Omega^q$  is a sheaf on  $\mathrm{Sm}_h/X$ . There is a direct image  $\gamma_*$  from sheaves on  $\mathrm{Sm}_h/X$  to sheaves on the small Zariski site  $X_{\mathrm{Zar}}$ .

Unfortunately we cannot directly apply Verdier's work on Čech cohomology of hypercovers. Comparing Čech and derived functor cohomology in this situation requires finite fiber products which don't exist in  $\mathrm{Sm}$ . However the standard comparison would show that

$$\underline{\Omega}_{X/\mathbb{C}}^\bullet \simeq R\gamma_* \Omega^\bullet$$

and thus

$$H_{dR}^i(X) \simeq \mathbf{H}_h^i(X, \Omega^\bullet) = \mathbf{H}_{\mathrm{Zar}}^i(X, R\gamma_* \Omega^\bullet)$$

giving our main result. (By GAGA [14] and results of [10], this would be isomorphic to its analytic counterpart.)

According to Jardine ([17]), hypercovers are just (semi-)representable local acyclic fibrations. Keeping this in mind, we generalize Verdier's work on Čech cohomology to arbitrary local acyclic fibrations of simplicial presheaves. The precise statement proved in the first section is

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**Theorem 2.13.** Let  $X_\bullet$  be a simplicial presheaf, and  $hD(X_\bullet)$  the homotopy category of local acyclic fibrations  $K_\bullet \rightarrow X_\bullet$ . Then for a bounded below complex of sheaves of abelian groups  $\mathcal{F}^\bullet$  with the filtration bête there is an isomorphism

$$\varinjlim_{K_\bullet \in hD(X_\bullet)} H^p(\mathrm{Tot} \mathrm{Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet)) \simeq \mathrm{Ext}^p(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet)$$

and there is a filtered quasi-isomorphism of ind-objects in the derived category

$$\varinjlim_{K_\bullet \in hD(X_\bullet)} \mathrm{Tot} \mathrm{Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet) \simeq R\mathrm{Hom}(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet).$$

Note the lack of hypotheses on fiber products in the underlying topos.

In practice one usually wants to restrict to local acyclic fibrations which satisfy some representability hypothesis. Define a semi-representable presheaf to be a presheaf that is isomorphic to a coproduct of representable presheaves; that one can restrict to semi-representable presheaves is an easy corollary of the above theorem. To satisfy stronger hypotheses than semi-representability seems to require something from the underlying topos – in our case we use the inclusion  $\mathrm{Sm}_h \subset \mathrm{Sch}_h$ .

Section three is occupied with “topological” matters. Using Du Bois’ results requires a comparison of the *ucd*- and *h*-topologies: after some preliminaries, we show every *h*-covering is a *ucd*-covering. We do not know of an example of a *ucd*-covering that is not an *h*-covering. Finally we show one can actually compute using representable presheaves in  $\mathrm{Sm}_h$ .

In section four we apply our work to the algebraic de Rham complex. Key in applying Du Bois’ results is Theorem 4.11, which compares *h*-hypercovers to Zariski hypercovers. This result comes from a generous suggestion of Alexander Beilinson. Also in this section is the proof that  $\Omega^q$  is a sheaf in the *h*-topology. These results with the Čech theory yield the main theorem.

This paper is based on my dissertation, and I owe thanks to the many people who helped me. Everything here has benefited from the guiding hand of my advisor, Madhav Nori, to whom I give my sincerest thanks. Alexander Beilinson has also provided invaluable help and advice. I would also like to thank Andrew Blumberg and Minhea Popa for stimulating mathematical discussions.

## 2. A GENERALIZED VERDIER THEOREM

**2.1. Local acyclic fibrations.** Let  $C$  be a site,  $\mathrm{Pre}C$  the category of presheaves of sets on  $C$ ,  $\mathrm{Sh}C$  the category of sheaves of sets on  $C$ , and  $s\mathrm{Pre}C, s\mathrm{Sh}C$  the categories of simplicial presheaves and sheaves. Note that, unlike [12, ex V 7.3.0], we do not assume the existence of products and finite fiber products in our site  $C$ . Let  $e$  be the terminal object of  $\mathrm{Sh}C$ . For a presheaf  $K$ , let  $\mathbb{Z}K$  denote the associated sheaf of free abelian groups; for a simplicial presheaf  $K_\bullet$ , let  $\mathbb{Z}K^\bullet$  denote the associated negative cochain complex of sheaves of free abelian groups; define  $\mathbb{Z} := \mathbb{Z}e$  the sheaf of free abelian groups associated to the terminal object  $e$ .

**Definition 2.1** (cf. [17, 6, 5]).

- (1) Let  $f : L_\bullet \rightarrow K_\bullet$  be a morphism of simplicial presheaves.  $f$  is called a *local acyclic fibration* if, for every  $U \in C$ , integer  $k \geq 0$  and diagram

$$\begin{array}{ccc} \partial\Delta^k & \longrightarrow & L_\bullet(U) \\ \downarrow & & \downarrow f(U) \\ \Delta^k & \longrightarrow & K_\bullet(U) \end{array}$$

there is a refinement (a covering sieve)  $R$  of  $U$  so that for every  $V \rightarrow U \in R$  there is a lift

$$\begin{array}{ccccc} \partial\Delta^k & \longrightarrow & L_\bullet(U) & \longrightarrow & L_\bullet(V) \\ \downarrow & & \downarrow f(U) & \nearrow f(V) & \downarrow f(V) \\ \Delta^k & \longrightarrow & K_\bullet(U) & \longrightarrow & K_\bullet(V) \end{array}$$

indicated by the dashed arrow. We say  $f$  satisfies the *local right lifting property* for the inclusion  $\partial\Delta^k \rightarrow \Delta^k$ .

- (2) For a presheaf  $M \in \text{Pre}C$ , let  $M_\bullet$  be the constant simplicial presheaf associated to  $M$ . We abuse notation and call the augmented simplicial presheaf  $K_\bullet \rightarrow M$  a local acyclic fibration if the morphism of simplicial presheaves  $K_\bullet \rightarrow M_\bullet$  is a local acyclic fibration.
- (3) Recall a simplicial presheaf is *semi-representable* if its components are isomorphic to coproducts of representable presheaves. A local acyclic fibration  $L_\bullet \rightarrow K_\bullet$  is a *hypercov* if both  $L_\bullet$  and  $K_\bullet$  are semi-representable.

Compare the following with [12, ex V Lemma 7.3.6]:

**Lemma 2.2.** *A morphism  $f : L_\bullet \rightarrow K_\bullet$  is a local acyclic fibration if and only if for every  $P_\bullet \hookrightarrow Q_\bullet$  an inclusion of constant simplicial sets with only finitely many non-degenerate simplices, we can locally lift diagrams*

$$\begin{array}{ccccc} P_\bullet & \longrightarrow & L_\bullet(U) & \longrightarrow & L_\bullet(V) \\ \downarrow & & \downarrow f(U) & \nearrow f(V) & \downarrow f(V) \\ Q_\bullet & \longrightarrow & K_\bullet(U) & \longrightarrow & K_\bullet(V) \end{array}$$

*Proof.* Induction on the definition. □

Recall a morphism of presheaves  $F \rightarrow G$  is a *covering morphism* if the associated morphism of sheaves is an epimorphism (see [11, II.5.2].)

*Remark 2.3.* For a morphism  $f : L_\bullet \rightarrow K_\bullet$ , Verdier uses the following equivalent definition of local acyclic fibration (which he calls “special”):

- (1) For each integer  $k \geq 0$ , the morphism  $\phi_k$  in the diagram is a covering morphism:

$$\begin{array}{ccc}
 L_k & \xrightarrow{f_k} & K_k \\
 & \searrow \phi_k & \nearrow \\
 & P_k & \\
 & \swarrow & \searrow \\
 (\text{cosk}_{k-1} L)_k & \xrightarrow{(\text{cosk}_{k-1} f)_k} & (\text{cosk}_{k-1} K)_k
 \end{array}$$

The vertical arrows are the coskeleton adjunction maps and  $P_k$  is the fiber product of  $K_k$  and  $(\text{cosk}_{k-1} L)_k$  by the arrows in the diagram.

- (2) The morphism  $f_0 : L_0 \rightarrow K_0$  is a covering morphism.

For a simplicial set  $S_\bullet$ , let  $\underline{S}_\bullet$  denote the associated constant simplicial presheaf.

**Proposition 2.4** ([6, Proposition 7.2]). *Let  $f : L_\bullet \rightarrow K_\bullet$  be a morphism of simplicial presheaves. Then the following are equivalent:*

- (1)  $f$  is a local acyclic fibration.
- (2) For every integer  $k \geq 0$ , the morphism

$$\mathcal{H}om(\underline{\Delta}^k, L_\bullet) \rightarrow \mathcal{H}om(\partial \underline{\Delta}^k, L_\bullet) \times_{\mathcal{H}om(\partial \underline{\Delta}^k, K_\bullet)} \mathcal{H}om(\underline{\Delta}^k, K_\bullet)$$

induced by the inclusion  $\partial \Delta^k \rightarrow \Delta^k$  and  $f$  is a covering morphism.

- (3)  $f$  is special in the sense of Verdier.

*Proof.*  $1 \Leftrightarrow 2$  is by definition. To show  $2 \Leftrightarrow 3$ , apply the isomorphisms  $X_k = \text{Hom}(\Delta^k, X_\bullet)$  and the coskeleton-skeleton adjunction to the covering condition

$$L_k \rightarrow (\text{cosk}_{k-1} L)_k \times_{(\text{cosk}_{k-1} K)_k} K_k$$

noting that  $\text{sk}_{k-1} \underline{\Delta}^k = \partial \underline{\Delta}^k$ . □

We recall the following basic results.

**Proposition 2.5** ([17, Proposition 2.9]). *If  $f : L_\bullet \rightarrow K_\bullet$  is a local acyclic fibration of simplicial presheaves, then the induced map  $\mathbb{Z}L^\bullet \rightarrow \mathbb{Z}K^\bullet$  is a quasi-isomorphism of complexes of sheaves.*

**Lemma 2.6** ([12, ex V Lemma 7.3.4]).

- (1) The composition of two local acyclic fibrations is a local acyclic fibration.
- (2) Local acyclic fibrations are preserved under base change.
- (3) Suppose  $K_\bullet$  is a generalized hypercover,  $f : L_\bullet \rightarrow M_\bullet$  a local acyclic fibration, and  $K_\bullet \rightarrow M_\bullet$  a morphism. Then the fiber product  $L_\bullet \times_{M_\bullet} K_\bullet$  is a generalized hypercover.

**2.2. Computing Ext.** Before proving our main theorem we require the following technical lemma.

**Lemma 2.7** (Lemma on computing Ext). *Let  $C$  be an abelian category with enough injectives,  $X^\bullet \in \text{Ch}^-(C)$  a fixed negative cochain complex, and  $G^\bullet \in \text{Ch}^+(C)$  a fixed positive cochain complex. Suppose  $D \subset \text{Ch}^-(C)/X^\bullet$  is a subcategory of the category of negative cochain complexes of  $C$  over  $X^\bullet$  with the following properties:*

- (1) The homotopy category  $hD$  (morphisms up to chain homotopy) is cofiltered.
- (2) For every complex  $K^\bullet \in D$ , object  $M \in C$ , and epimorphism  $u : M \rightarrow K^n$ , there is a complex  $L^\bullet \in D$  and a morphism  $f : L^\bullet \rightarrow K^\bullet$  whose degree  $n$  part factors as

$$f^n : L^n \rightarrow M \xrightarrow{u} K^n.$$

- (3) Every  $K^\bullet \in D$  has structure morphism  $K^\bullet \rightarrow X^\bullet$  a quasi-isomorphism.

Then there is an isomorphism of functors

$$\varinjlim_{K^\bullet \in hD} H^p(\text{Tot Hom}_C(K^\bullet, G^\bullet)) \simeq \text{Ext}_C^p(X^\bullet, G^\bullet).$$

Here  $\text{Ext}$  is hyper- $\text{Ext}$ .

Some explanation:

- (1) We first work with the case when  $G^\bullet = G$  a single object concentrated in degree zero. We compute  $\text{RHom}(K^\bullet, G)$  by taking an injective resolution  $I^\bullet$  of  $G$ , yielding a first quadrant double complex  $\text{Hom}_C(K^\bullet, I^\bullet)$  (giving  $\text{Hom}_C(K^a, I^b)$  bidegree  $(-a, b)$ ) which has total complex  $\text{Tot Hom}_C(K^\bullet, I^\bullet)$ . This complex has a decreasing filtration by columns

$$F^l \text{Tot}^m \text{Hom}_C(K^\bullet, I^\bullet) = \bigoplus_{\substack{-a+b=m \\ a \leq l}} \text{Hom}_C(K^a, I^b).$$

We get a first quadrant convergent spectral sequence

$$\begin{aligned} E_1^{p,q} &= H^q(\text{Hom}_C(K^p, I^\bullet)) = \text{Ext}_C^q(K^p, G) \\ &\Rightarrow H^{p+q} \text{RHom}(K^\bullet, G) = \text{Ext}_C^{p+q}(X^\bullet, G). \end{aligned}$$

- (2) Since only  $hD$  is cofiltered, it does not make sense to take the filtered colimit of the  $E_1$  terms over  $hD$ . However, since the  $E_2$  terms are the horizontal cohomology of the  $E_1$  terms and chain homotopic maps induce the same map on cohomology, we can take the filtered colimit of the  $E_2$  terms over  $hD$ . This yields a limit spectral sequence

$$E_2^{p,q} = \varinjlim_{K \in hD} H^p(\text{Ext}_C^q(K^\bullet, G)) \Rightarrow \text{Ext}_C^{p+q}(X^\bullet, G).$$

The objects on the left hand side are cohomologies of the complexes of  $\text{Ext}_C^q$  by varying the  $K^\bullet$ . The contention of the theorem is that the terms with  $q > 0$  vanish in the limit, collapsing the spectral sequence at the  $E_2$  page, yielding an isomorphism

$$\varinjlim_{K \in hD} H^p(\text{Hom}_C(K^\bullet, G)) = \text{Ext}_C^p(X^\bullet, G).$$

- (3) Using property 2 of  $D$ , the remarks show it is enough to prove the following well-known lemma.

**Lemma 2.8.** *Let  $C$  be an abelian category with enough injectives. For any  $K, A \in C$ , any  $q > 0$  and any extension class*

$$\gamma \in \text{Ext}_C^q(K, A)$$

*there is an epimorphism  $f : M \rightarrow K$  so that  $f^*(\gamma) = 0$  in  $\text{Ext}_C^q(M, A)$ .*

*Proof.* For a fixed  $q > 0$  choose a truncated injective resolution

$$0 \rightarrow A \rightarrow I^0 \rightarrow \cdots \rightarrow I^{q-1} \rightarrow J \rightarrow 0$$

where  $J$  is the cokernel of  $I^{q-2} \rightarrow I^{q-1}$ . Applying the functor  $\text{Hom}(K, \_)$  yields the complex

$$\text{Hom}(K, I^0) \rightarrow \cdots \rightarrow \text{Hom}(K, I^{q-1}) \rightarrow \text{Hom}(K, J) \rightarrow \text{Ext}^q(K, A) \rightarrow 0.$$

Lift  $\gamma$  to a homomorphism  $\sigma : K \rightarrow J$  in  $\text{Hom}(K, J)$ . Form the fiber product  $I^{q-1} \times_J K$  using  $\sigma$ . The natural projection map  $f : M = I^{q-1} \times_K K \rightarrow K$  yields a map of complexes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \text{Hom}(K, I^{q-1}) & \longrightarrow & \text{Hom}(K, J) & \longrightarrow & \text{Ext}^q(K, A) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \text{Hom}(M, I^{q-1}) & \longrightarrow & \text{Hom}(M, J) & \longrightarrow & \text{Ext}^q(M, A) \longrightarrow 0. \end{array}$$

By construction,  $f^*(\sigma) \in \text{Hom}(M, J)$  is  $u \circ p_2$  in the cartesian square

$$\begin{array}{ccc} I^{q-1} \times_K M & \xrightarrow{p_2} & M \\ p_1 \downarrow & & \downarrow u \\ I^{q-1} & \xrightarrow{c} & K \end{array}.$$

But  $u \circ p_2 = c \circ p_1$  is the image of  $p_1 \in \text{Hom}(M, I^{q-1})$ . Hence  $f^*(\sigma)$  is a coboundary and so  $f^*(\gamma)$  is zero.  $\square$

- (4) Let  $f : I \rightarrow D^+(C)$  be a filtered system in the derived category of  $C$ . The associated ind-object is denoted by

$$\varinjlim_{M^\bullet \in f(I)} M^\bullet.$$

We define the cohomology of this ind-object by the equation

$$H^k \left( \varinjlim_{M^\bullet \in f(I)} M^\bullet \right) := \varinjlim_{M^\bullet \in f(I)} H^k(M^\bullet).$$

We note that, in the case where the ind-object is representable, this agrees with the cohomology of the limit object since  $H^k(\varinjlim M^\bullet) = \varinjlim H^k(M^\bullet)$ , cf. [18, 1.12.7], using the model of the derived category via injectives (as in [7, III.5.22].) We say a map of ind-objects is a quasi-isomorphism if it induces an isomorphism on cohomology.

**Corollary 2.9.** *Suppose  $D \subset \text{Ch}^-(C)$  satisfies the hypotheses of Lemma 2.7. Then there is a natural quasi-isomorphism*

$$\varinjlim_{K^\bullet \in {}^h D} \text{Hom}_C(K^\bullet, G) \simeq R\text{Hom}_C(X^\bullet, G).$$

- (5) The results extend to complexes concentrated in a single non-zero degree, by reindexing.

(6) Now let  $G^\bullet \in \text{Ch}^b(C)$  be a finite complex. By the corollary we see that

$${}^{\textstyle \varinjlim}_{K^\bullet \in hD} \text{Hom}_C(K^\bullet, \_)$$

takes short exact sequences to exact triangles. If  $G^\bullet$  is a bounded complex, it has a finite truncation filtration with subquotients complexes concentrated in a single degree. This gives the result for finite complexes.

(7) For a bounded below complex  $G^\bullet$ , we note that

$$\varinjlim_n R\text{Hom}(K^\bullet, G^{\leq n}) = R\text{Hom}(K^\bullet, \varinjlim_n G^{\leq n}) = R\text{Hom}(K^\bullet, G^\bullet)$$

if  $K^\bullet$  is a bounded above complex:  $R^i \text{Hom}(K^\bullet, G^\bullet) = R^i \text{Hom}(K^\bullet, G^{\leq n})$  for some  $n$  sufficiently large, since the overlap between  $K^\bullet$  and  $G^\bullet[i]$  is finite. Likewise

$$\varinjlim_n {}^{\textstyle \varinjlim}_{K^\bullet \in hD} \text{Hom}_C(K^\bullet, G^{\leq n}) = {}^{\textstyle \varinjlim}_{K^\bullet \in hD} \text{Hom}_C(K^\bullet, \varinjlim_n G^{\leq n}).$$

This gives the result for bounded below complexes, and thus completes the proof of Lemma 2.7.

**Corollary 2.10.** *Suppose  $D \subset \text{Ch}^-(C)$  satisfies the hypotheses of Lemma 2.7, and  $G^\bullet \in \text{Ch}^+(C)$  is a fixed bounded below complex. Then there is a natural filtered quasi-isomorphism*

$${}^{\textstyle \varinjlim}_{K^\bullet \in hD} \text{Tot Hom}_C(K^\bullet, G^\bullet) \simeq R\text{Hom}_C(X^\bullet, G^\bullet)$$

where on each side the filtration arises from the filtration *bête* on  $G^\bullet$ .

**2.3. Main theorem.** Recall  $C$  is a site, possibly without finite products and fiber products.

**Definition 2.11.** For a fixed simplicial presheaf  $X_\bullet \in s\text{Pre } C$ , let  $D(X_\bullet)$  denote the subcategory of  $s\text{Pre } C/X_\bullet$  of local acyclic fibrations  $K_\bullet \rightarrow X_\bullet$ .

For any category of simplicial objects  $E$ , write  $hE$  to be the same category with morphisms up to simplicial homotopy. In general this is not an equivalence relation, we use the relation generated by simplicial homotopy.

**Proposition 2.12** (cf. [12, ex V Theorem 7.3.2]).

*Fix a simplicial presheaf  $X_\bullet \in s\text{Pre } C$ .*

- (1) *The homotopy category  $hD(X_\bullet)$  is cofiltered.*
- (2) *For every  $K_\bullet \in hD(X_\bullet)$ , object  $M \in C$ , and covering morphism  $u : M \rightarrow K_n$ , there is an object  $L_\bullet \in hD(X_\bullet)$  and a morphism  $f : L_\bullet \rightarrow K_\bullet$  whose degree  $n$  part factors as*

$$f_n : L_n \rightarrow M \xrightarrow{u} K_n.$$

- (3) *For every  $K_\bullet \in D(X_\bullet)$ , the structure morphism  $K_\bullet \rightarrow X_\bullet$  induces a quasi-isomorphism*

$$\mathbb{Z}K^\bullet \rightarrow \mathbb{Z}X^\bullet.$$

*Proof.*

- Proof of part 3 of Proposition 2.12:  
This is just Proposition 2.5.

- Proof of part 2 of Proposition 2.12:

The following is mostly unchanged from Verdier's original.

Let  $j_{n*}$  the right adjoint of “taking the degree  $n$  component.” I claim  $j_{n*}$  takes covering morphisms to local acyclic fibrations. Let  $f : A \rightarrow B$  be a covering morphism of presheaves. Then we must check, for an open  $U \in C$ , that we can locally lift a diagonal in a diagram

$$\begin{array}{ccc} \partial\Delta_n^k & \longrightarrow & A(U) \\ \downarrow & & \downarrow \\ \Delta_n^k & \longrightarrow & B(U). \end{array}$$

But since  $A \rightarrow B$  is a covering, it is a surjection after a refinement  $V$  of  $U$ , so we can always lift  $\Delta_n^k \rightarrow A(V)$ .

To prove part 2, form the cartesian diagram

$$\begin{array}{ccc} L_{\bullet} & \longrightarrow & j_{n*}M \\ \downarrow & & \downarrow \\ K_{\bullet} & \longrightarrow & j_{n*}j_n^*K_{\bullet} = j_{n*}K_n \\ \downarrow & & \\ X_{\bullet} & & \end{array}$$

where the right vertical arrow is given by functoriality and the bottom horizontal arrow is given by adjunction. The right vertical arrow is a local acyclic fibration by the above remark. By Lemma 2.6  $L_{\bullet} \rightarrow K_{\bullet} \rightarrow X_{\bullet}$  is a local acyclic fibration, and  $L_n \rightarrow K_n$  factors as  $L_n \rightarrow M \rightarrow K_n$ .

- Proof of part 1 of Proposition 2.12: <sup>1</sup>

Suppose we are given a diagram

$$\begin{array}{ccc} & & A_{\bullet} \\ & & \downarrow \\ B_{\bullet} & \longrightarrow & K_{\bullet} \end{array}$$

in  $D(X_{\bullet})$ . Set  $L_{\bullet} = A_{\bullet} \times_{X_{\bullet}} B_{\bullet}$  which exists in  $s\text{Pre}C$ . Lemma 2.6 shows the canonical map  $L_{\bullet} \rightarrow X_{\bullet}$  is a local acyclic fibration, so it is in  $D(X_{\bullet})$ . This gives a possibly non-commutative diagram

$$\begin{array}{ccccc} L_{\bullet} & & \longrightarrow & & A_{\bullet} \\ \downarrow & & & \searrow & \downarrow \\ & & & K_{\bullet} & \\ \downarrow & \nearrow & & \searrow & \downarrow \\ B_{\bullet} & & \longrightarrow & & X_{\bullet} \end{array}$$

Hence we have two maps  $L_{\bullet} \rightrightarrows K_{\bullet}$  which we wish to equalize up to homotopy. Thus to prove  $hD(X_{\bullet})$  is cofiltered, it is enough to show that for

<sup>1</sup>We warn the reader that there is a small, inconsequential error in Verdier's original.



every pair of morphisms in  $D(X_\bullet)$

$$L_\bullet \begin{matrix} \xrightarrow{u_0} \\ \xrightarrow{u_1} \end{matrix} K_\bullet$$

there is a morphism  $v : M_\bullet \rightarrow L_\bullet$  in  $D(X_\bullet)$  so that the two morphisms  $u_0v$  and  $u_1v$  are homotopic, i.e. there are commutative diagrams

$$\begin{array}{ccc} M_\bullet & \xrightarrow{e_i} & M_\bullet \times \underline{\Delta}^1 \\ v \downarrow & & \downarrow w \\ L_\bullet & \xrightarrow{u_i} & K_\bullet \end{array}$$

for  $i = 0, 1$ , where the  $e_i$  are the standard inclusions, and  $w$  is the homotopy.

The set of such diagrams for fixed  $M_\bullet$  and  $L_\bullet \begin{matrix} \xrightarrow{u_0} \\ \xrightarrow{u_1} \end{matrix} K_\bullet$  is given by

$$\mathrm{Hom}(M_\bullet \times \underline{\Delta}^1, K_\bullet) \times_{\mathrm{Hom}(M_\bullet, K_\bullet \times K_\bullet)} \mathrm{Hom}(M_\bullet, L_\bullet)$$

where the map from  $\mathrm{Hom}(M_\bullet, L_\bullet)$  to  $\mathrm{Hom}(M_\bullet, K_\bullet \times K_\bullet)$  is induced from  $u_1 \times u_2$ , and the map  $\mathrm{Hom}(M_\bullet \times \underline{\Delta}^1, K_\bullet)$  to  $\mathrm{Hom}(M_\bullet, K_\bullet \times K_\bullet)$  is induced from  $e_0 \times e_1$ .

The functor

$$\mathrm{Hom}(\_\times \underline{\Delta}^1, K_\bullet) \times_{\mathrm{Hom}(\_, K_\bullet \times K_\bullet)} \mathrm{Hom}(\_, L_\bullet)$$

is equal to

$$\begin{aligned} & \mathrm{Hom}(\_, s\mathcal{H}om(\underline{\Delta}^1, K_\bullet)) \times_{\mathrm{Hom}(\_, K_\bullet \times K_\bullet)} \mathrm{Hom}(\_, L_\bullet) \\ &= \mathrm{Hom}(\_, s\mathcal{H}om(\underline{\Delta}^1, K_\bullet) \times_{K_\bullet \times K_\bullet} L_\bullet) \end{aligned}$$

and so is representable. Call this representing object  $F_\bullet$ . We must show that  $F_\bullet \rightarrow X_\bullet$  is a local acyclic fibration.  $F$  is the pullback in the square in the diagram

$$\begin{array}{ccccc} F_\bullet & \xrightarrow{\quad} & s\mathcal{H}om(\underline{\Delta}^1, K_\bullet) & & \\ \downarrow & & \downarrow & & \\ L_\bullet & \xrightarrow{d} & L_\bullet \times L_\bullet & \xrightarrow{u_0 \times u_1} & K_\bullet \times K_\bullet \\ & & & \searrow & \\ & & & & X_\bullet \times X_\bullet \\ & & & & \searrow \begin{matrix} p_1 \\ p_2 \end{matrix} \\ & & & & X_\bullet \end{array}$$

where  $d$  is the diagonal. Note all maps to  $X_\bullet$  are the same, and

$$K_\bullet \times K_\bullet = s\mathcal{H}om(\underline{\partial\Delta}^1, K_\bullet).$$

Thus to lift a diagram

$$\begin{array}{ccc} \partial\Delta^k & \longrightarrow & F_\bullet(U) \\ \downarrow & & \downarrow \\ \Delta^k & \longrightarrow & X_\bullet(U) \end{array}$$

we have to lift from  $X_\bullet$  to  $L_\bullet$ ,  $K_\bullet \times K_\bullet$  and  $s\mathcal{H}om(\underline{\Delta}^1, K_\bullet)$  with the following compatibility condition: the lift  $\Delta^k \rightarrow L_\bullet(V)$  yields by composition with the diagonal a lift  $\Delta^k \rightarrow L_\bullet(V) \times L_\bullet(V)$ , or a map  $\Delta^k \times \partial\Delta^1 \rightarrow L_\bullet(V)$ . By composition with the map  $u_0 \times u_1$  we get a lift  $\Delta^k \times \partial\Delta^1 \rightarrow K_\bullet(V)$ . Meanwhile a lift to  $s\mathcal{H}om(\underline{\Delta}^1, K_\bullet)$  is a map  $\Delta^k \times \Delta^1 \rightarrow K(V)$ , which by pre-composition with the inclusion  $\partial\Delta^1 \subset \Delta^1$  yields a map  $\Delta^k \times \partial\Delta^1 \rightarrow K(V)$ . We require these two maps are equal.

But we can guarantee this as follows: giving the lifting diagram above, we extend by projection to the first factor to a diagram

$$\begin{array}{ccccccc} \partial\Delta^k \times \partial\Delta^1 & \longrightarrow & \partial\Delta^k & \longrightarrow & F_\bullet(V) & \longrightarrow & L_\bullet(V) \\ \downarrow & & \downarrow & \nearrow \text{dashed} & & & \downarrow \\ \Delta^k \times \Delta^1 & \longrightarrow & \Delta^k & \longrightarrow & X_\bullet(V) & & \end{array}$$

By Lemma 2.2 we can lift to get the dashed arrow. This yields a composition

$$\Delta^k \times \partial\Delta^1 \rightarrow \Delta^k \times \Delta^1 \rightarrow L_\bullet(V) \rightarrow K_\bullet(V)$$

e.g. lifts to  $L_\bullet(V) \times L_\bullet(V)$  and  $s\mathcal{H}om(\underline{\Delta}^1, K_\bullet)(V)$  which map to the same lift to  $K_\bullet(V) \times K_\bullet(V)$ . The compatibility of the maps to  $X_\bullet$ , and the fact that  $L_\bullet \rightarrow L_\bullet \times L_\bullet$  is the diagonal, ensures that these lifts are compatible with the maps in the fiber product.

□

Fix a  $X_\bullet \in s\text{Pre } C$ . Let  $\text{Ab}(\text{Sh } C)$  be the category of sheaves of abelian groups on  $C$ . A simplicial presheaf  $K_\bullet$  yields a negative cochain complex of sheaves of free abelian groups  $\mathbb{Z}K^\bullet$ . We abuse notation and also call  $D(X_\bullet)$  the image of  $D(X_\bullet)$  inside  $\text{Ch}^-(\text{Ab}(\text{Sh } C))$  under this functor. Note that simplicial homotopy of simplicial presheaves becomes chain homotopy of cochain complexes under this functor.

Our basic result on hyper-Čech cohomology is

**Theorem 2.13.** *Let  $X_\bullet$  be a simplicial presheaf, and  $hD(X_\bullet)$  the homotopy category of local acyclic fibrations  $K_\bullet \rightarrow X_\bullet$ . Then for a bounded below complex of sheaves of abelian groups  $\mathcal{F}^\bullet$  with the filtration bête*

$$\varinjlim_{K_\bullet \in hD(X_\bullet)} H^p(\text{Tot Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet)) \simeq \text{Ext}^p(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet)$$

and there is a filtered quasi-isomorphism of ind-objects in the derived category

$$\varinjlim_{K_\bullet \in hD(X_\bullet)} \text{Tot Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet) \simeq R\text{Hom}(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet).$$

*Proof.* According to Proposition 2.12,  $D(X_\bullet)$  is a subcategory of  $\text{Ch}^-(\text{Ab}(\text{Sh } C))$  which satisfies the properties of the Lemma 2.7, the lemma on computing  $\text{Ext}$ , which gives the result. For the last part, apply Corollary 2.10. □

#### 2.4. Semi-representability and finite representability.

**Definition 2.14.** A presheaf is *semi-representable* if it is isomorphic to a coproduct of representable presheaves. A presheaf is *finitely representable* if it is isomorphic to a finite coproduct of representable presheaves. A simplicial presheaf is semi-representable (resp. finitely representable) if all its components are.

The theorems above show representability hypotheses are not important in the computation of sheaf cohomology. However typically one wishes to compute with representable or semi-representable presheaves. For this we have

**Lemma 2.15** (A Godement-type lemma). *Any presheaf is covered by a semi-representable presheaf.*

*Proof.* For a presheaf  $F$ , we have the presheaf surjection

$$\coprod_{(X \in C, s \in F(X))} h_X \rightarrow F$$

where  $h_X$  denotes the representable presheaf given by  $\text{Hom}(\_, X)$ . Since  $\text{Hom}(X, F) = F(X)$  the morphism is given by  $s$ . This is obviously surjective on the level of sets, and since sheafification is exact, is a covering.  $\square$

*Remark 2.16.* We make use of the formalism of *split simplicial objects*, cf. [12, ex Vbis 5.1] or [3, 6.2.2], which allows us to construct semi-representable simplicial presheaves inductively by only specifying the non-degenerate pieces. The degeneracies are satisfied by adding copies of the lower degree pieces; all maps between such objects are isomorphisms, so will satisfy whatever requirements we have of them (properness, coverings, et cetera) and will come equipped inductively via the degeneracies with maps to any desired target.

**Proposition 2.17.** *Let  $SR(X_\bullet)$  be the full subcategory of  $D(X_\bullet)$  of objects whose components are semi-representable. Then  $hSR(X_\bullet)$  is a cofinal subcategory of  $hD(X_\bullet)$ . Thus for a bounded below complex of sheaves of abelian groups  $\mathcal{F}^\bullet$  with the filtration bête*

$$\varinjlim_{K_\bullet \in hSR(X_\bullet)} H^p(\text{Tot Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet)) \simeq \text{Ext}^p(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet)$$

*and there is a filtered quasi-isomorphism of ind-objects in the derived category*

$$\varinjlim_{K_\bullet \in hSR(X_\bullet)} \text{Tot Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet) \simeq R\text{Hom}(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet).$$

*Proof.* It is enough to show, for any local acyclic fibration  $K_\bullet \rightarrow X_\bullet$ , there is a local acyclic fibration  $L_\bullet \rightarrow K_\bullet$  with  $L_\bullet$  semi-representable. We construct one inductively as follows: set  $L_0 \rightarrow K_0$  a semi-representable cover given by the Godement lemma. Having constructed  $L_\bullet$  to degree  $i-1$ , set

$$L' \rightarrow (i_{i-1*} L)_i \times_{(\text{cosk}_{i-1} K)_i} K_i$$

to be a semi-representable cover given by the Godement lemma. We set  $L_i$  to be the union of  $L'$  and the copies of the  $L_k$  for  $k < i$  needed to satisfy the degeneracy relations; see Remark 2.16.  $\square$

*Remark 2.18.* This gives a generalized version of Verdier's theorem on hypercovers ([12, ex V Theorem 7.4.1].)

On sites without finite products and fiber products, we need some additional hypotheses for finite representability. The following result will be useful in application to  $\text{Sm}_h \subset \text{Sch}_h$ , cf. Corollary 3.18.

**Proposition 2.19.** *Suppose the site  $C$  is full subcategory of a larger site  $C'$ , and*

- (1) *The topology on  $C'$  is generated by a pretopology all of whose covering families are finite.*
- (2)  *$C$  has the induced topology.*
- (3)  *$C'$  has finite products and fiber products.*
- (4) *Every  $Y \in C'$  can be covered by an  $X \in C$ .*

*Let  $X_\bullet \in s\text{Pre}C'$  be a finitely representable simplicial presheaf. Let  $\text{FR}_C(X_\bullet)$  be the subcategory of  $\text{SR}_{C'}(X_\bullet) = \{\text{semi-representable local acyclic fibrations in } C'\}$  whose components are finitely representable and in  $C$ . Then  $h\text{FR}_C(X_\bullet)$  is cofinal in  $h\text{SR}_{C'}(X_\bullet)$ . Thus for a bounded below complex of sheaves of abelian groups  $\mathcal{F}^\bullet$  with the filtration *bête**

$$\varinjlim_{K_\bullet \in h\text{FR}_C(X_\bullet)} H^p(\text{Tot Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet)) \simeq \text{Ext}^p(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet)$$

*and there is a filtered quasi-isomorphism of ind-objects in the derived category*

$$\varinjlim_{K_\bullet \in h\text{FR}_C(X_\bullet)} \text{Tot Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet) \simeq R\text{Hom}(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet).$$

*Proof.* The hypotheses on  $C$  and  $C'$  show that

- (1) Every covering morphism  $F \rightarrow G$  in  $C'$  where  $F$  is semi-representable and  $G$  is finitely representable can be refined  $E \rightarrow F \rightarrow G$  where  $E$  is finitely representable in  $C$  and  $E \rightarrow G$  is a covering morphism.
- (2) Finite limits of finitely representable presheaves in  $C'$  can be covered by finitely representable presheaves in  $C$ .

By Verdier's theorem it is enough to show, for every semi-representable local acyclic fibration  $K_\bullet \rightarrow X_\bullet$  in  $C'$ , there is a finitely representable local acyclic fibration  $L_\bullet \rightarrow X_\bullet$  with  $L_\bullet$  in  $C$  and a map over  $X_\bullet$  of simplicial presheaves  $L_\bullet \rightarrow K_\bullet$ . Set  $L_0 \subset K_0$  a subpresheaf which is finitely representable in  $C$  and covers  $X_0$ . Suppose inductively we have constructed  $L_\bullet$  to degree  $i-1$ . Then  $(i_{i-1} \ast L)_i \times_{(\text{cosk}_{i-1} X)_i} X_i$  is a finite limit of finitely representable presheaves, so cover it with  $L'$  a finitely representable in  $C$ . Construct the fiber product  $L' \times_{(\text{cosk}_{i-1} X)_i} K_i$ , cover it with a semi-representable  $L''$ . Then  $L'' \rightarrow L'$  is a cover of a finitely representable by a semi-representable, so take  $L''' \rightarrow L'' \rightarrow L'$  with  $L'''$  finitely representable in  $C$  and  $L''' \rightarrow L'$  a cover. As before we have to add copies of  $L_k$  for  $k < i$  to satisfy degeneracy conditions, cf. Remark 2.16. By construction there is a map  $L_\bullet \rightarrow K_\bullet$  and the composite  $L_\bullet \rightarrow X_\bullet$  is a local acyclic fibration.  $\square$

### 3. $h$ - AND $ucd$ -TOPOLOGIES

#### 3.1. The $h$ -topology.

**Definition 3.1.** A  $\mathbb{C}$ -scheme is a separated scheme finite type over the field of complex numbers. Let  $\text{Sch}$  denote the category of  $\mathbb{C}$ -schemes, and let  $\text{Sm} \subset \text{Sch}$  denote the full subcategory of smooth  $\mathbb{C}$ -schemes. If  $X \in \text{Sch}$ , let  $\text{Sch}/X, \text{Sm}/X$  denote the categories of  $\mathbb{C}$ -schemes and smooth  $\mathbb{C}$ -schemes over  $X$ .

We recall Voevodsky's ([22])  $h$ -topology:

**Definition 3.2.** A morphism  $f : X \rightarrow Y$  is called a *topological epimorphism* if the underlying morphism of topological spaces is a topological quotient map: it is surjective on sets and  $U \subset Y$  is open if and only if  $f^{-1}(U)$  is open in  $X$ . A *universal topological epimorphism*, or an  *$h$ -covering*, is a morphism  $X \rightarrow Y$  so that for any  $Z \rightarrow Y$ , the base change morphism

$$X \times_Z Y \rightarrow Z$$

is a topological epimorphism.

A useful necessary but not sufficient characterization of  $h$ -coverings is given by the following.

**Proposition 3.3** ([22, Proposition 3.1.3]). *Let  $f : X \rightarrow Y$  be a morphism of schemes, and  $X' \subset X$  the union of the irreducible components of  $X$  which dominates some component of  $Y$ . If  $f$  is an  $h$ -covering then  $f(X') = Y$ .*

**Definition 3.4.** The  $h$ -topology is the topology on  $\text{Sch}$  induced from the pretopology given by finite families  $\{U_i \rightarrow X\}$  where  $\coprod U_i \rightarrow X$  is an  $h$ -covering. We denote the site of  $\mathbb{C}$ -schemes with the  $h$ -topology  $\text{Sch}_h$ .  $\text{Sm}$  inherits a topology from  $\text{Sch}_h$  as in [11, ex III 3.1]; by resolution of singularities ([16, 2]) this topology is just given by restricting covering sieves of  $\text{Sch}_h$  to  $\text{Sm}$ ; we denote this site  $\text{Sm}_h$ .

*Remark 3.5.* Note that, by resolution of singularities ([16, 2]),  $\text{Sm}_h \subset \text{Sch}_h$  satisfy the conditions of Proposition 2.19, so finitely representable hypercovers compute sheaf cohomology in  $\text{Sm}_h$ .

**Theorem 3.6** ([11, ex III Theorem 4.1]). *Let  $C, C'$  be small categories,  $u : C \rightarrow C'$  a fully faithful functor. Suppose  $C'$  has a Grothendieck topology, and let  $C$  have the induced topology. If every object of  $C'$  can be covered by an object of  $C$ , then the functor  $F \mapsto F \circ u$  is an equivalence of the category of sheaves on  $C'$  with the category of sheaves on  $C$ .*

**Corollary 3.7.** *The category of sheaves on  $\text{Sm}_h$  is equivalent via the natural embedding to the category of sheaves on  $\text{Sch}_h$ .*

*Proof.* Resolution of singularities ([16, 2]) gives smooth  $h$ -coverings of arbitrary  $\mathbb{C}$ -schemes.  $\square$

### 3.2. Cohomological Descent.

**Definition 3.8** (Cohomological Descent). An augmented simplicial  $\mathbb{C}$ -scheme  $e : K_\bullet \rightarrow X$  is a *cohomological descent resolution* if the adjunction

$$\text{id}_{an} \rightarrow Re_{an*} e_{an}^*$$

is an isomorphism; here we use the analytic topology. According to [13, ex XVI 4.1], if one restricts to rational vector spaces, this is the same as requiring  $\mathbb{Q}_{l,X} \simeq Re_*(\mathbb{Q}_{l,K_\bullet})$  in the étale topology. The morphism  $e$  is a *universal cohomological descent resolution* (or a *ucd-resolution*) if it is a cohomological descent resolution after any base change.

A morphism of  $\mathbb{C}$ -schemes  $Y \rightarrow X$  is of *cohomological descent* if  $\text{cosk}_0(Y/X) \rightarrow X$  is a cohomological descent resolution (where  $\text{cosk}_0(Y/X)$  is the coskeleton functor in the category of schemes over  $X$ .) A morphism  $Y \rightarrow X$  is *universally of cohomological descent* (or a *ucd-cover*) if every base change is of cohomological descent.

Some basic results:

**Lemma 3.9** ([3, 5.3.5]). *A morphism with a local section is a ucd-covering. A proper surjection is a ucd-covering.*

**Lemma 3.10** ([3, 5.3.5]). (1) *The composition of ucd-coverings is a ucd-covering.*  
 (2) *If the composition  $X \rightarrow Y \xrightarrow{f} Z$  is a ucd-covering, then  $f$  is a ucd-covering.*

*Proof.* See [1, Theorem 7.5] for a proof.  $\square$

According to [3, 5.3.5] ucd-coverings form a pretopology on  $\text{Sch}$ . We deviate from Deligne, however, in taking the pretopology generated by only finite families  $\{U_i \rightarrow X\}$  where  $\coprod U_i \rightarrow X$  is a ucd-covering. (Deligne and Du Bois in practice use only representable simplicial objects so there is no difference.) We denote the topology generated by this pretopology the universal cohomological descent topology, or the ucd-topology.

Let  $\text{Sch}_{\text{ucd}}$  be the category of  $\mathbb{C}$ -schemes with the ucd-topology. Since resolution of singularities are ucd-coverings, by the exact same argument as for the  $h$ -topology, the induced topology on  $\text{Sm}$  (denoted  $\text{Sm}_{\text{ucd}}$ ) is given by restricting the covering sieves of  $\text{Sch}_{\text{ucd}}$ , and the categories of sheaves on  $\text{Sch}_{\text{ucd}}$  and  $\text{Sm}_{\text{ucd}}$  are equivalent.

*Remark 3.11.* Again  $\text{Sch}_{\text{ucd}}$  and  $\text{Sm}_{\text{ucd}}$  satisfy the conditions of Proposition 2.19, so finitely representable hypercovers compute sheaf cohomology in  $\text{Sm}_{\text{ucd}}$ .

The basic, almost circular theorem is

**Theorem 3.12** ([3, 5.3.5]). *Let  $e : K_\bullet \rightarrow X$  be a hypercover in the topology of universal cohomological descent. Then  $f$  is a universal cohomological descent resolution.*

*Remark 3.13.* Note that both the  $h$ -topology and the ucd-topology refer to an underlying topology: the  $h$ -topology refers to the Zariski topology, and the ucd-topology refers to the étale or analytic topologies.

### 3.3. Comparison of the $h$ - and ucd-topologies.

**Lemma 3.14** ([22, Theorem 3.1.9]). *An  $h$ -covering  $Y \rightarrow X$  of an excellent reduced noetherian scheme  $X$  can be refined  $Y' \rightarrow Y \rightarrow X$  to an  $h$ -covering of normal form:  $Y' \rightarrow X$  factors as  $s \circ f \circ i$  where  $i$  is an open covering,  $f$  is a finite surjective morphism, and  $s$  is a blowup of a closed subscheme.*

**Corollary 3.15.** *An  $h$ -covering  $Y \rightarrow X$  in  $\text{Sm}_h$  can be refined to  $Y' \rightarrow Y \rightarrow X$ , where  $Y' \rightarrow X$  factors into  $Y' \rightarrow Z \rightarrow X$ , where  $Y' \rightarrow Z$  is a Zariski open cover, and  $Z \rightarrow X$  is proper, and  $Y'$  and  $Z$  are smooth. Moreover, we may assume both  $Y'$  and  $Z$  are quasi-projective.*

*Proof.*  $\mathbb{C}$ -schemes are excellent. Factor  $Y \rightarrow X$  to  $Y'' \rightarrow Y \rightarrow X$  with  $Y'' \rightarrow Z' \rightarrow X$  where  $Y'' \rightarrow Z'$  is a Zariski open cover and  $Z' \rightarrow X$  is proper (composition of a finite morphism and a blowup.) Use resolution of singularities to get  $Z \rightarrow Z' \rightarrow X$  proper, take  $Y' = Y'' \times_{Z'} Z$ , which will be a Zariski open cover of  $Z$ .

To get the last statement, use Chow's lemma [8, 5.6.1, 5.6.2] to get  $Z' \rightarrow Z$  by a projective surjective morphism with  $Z'$  quasi-projective, and the base change  $Y'' = Y' \times_Z Z'$  is a Zariski open cover of  $Z'$ .  $\square$

**Corollary 3.16.** *An  $h$ -covering in  $\text{Sch}$  or  $\text{Sm}$  is a ucd-covering.*

*Proof.* By the lemma or the corollary an  $h$ -covering  $f : Y \rightarrow X$  in either  $\text{Sch}$  or  $\text{Sm}$  has a refinement  $Y' \xrightarrow{g} Y \xrightarrow{f} X$  in either  $\text{Sch}$  or  $\text{Sm}$  where  $f \circ g$  factors into a composition of morphisms which are universally of cohomological descent. Hence by Lemma 3.10  $f$  is universally of cohomological descent.  $\square$

By the above proposition, we have continuous functors  $\text{Sch}_h \rightarrow \text{Sch}_{ucd}$  and  $\text{Sm}_h \rightarrow \text{Sm}_{ucd}$  (see [11, ex III Proposition 1.6]) and thus geometric morphisms of their associated topoi of sheaves. We do not know of an example of a  $ucd$ -covering which is not an  $h$ -covering.

### 3.4. Representable hypercovers in the $h$ - or $ucd$ -topologies.

**Lemma 3.17.** *If  $L_\bullet$  is a finitely representable hypercover in either  $\text{Sm}_h, \text{Sch}_h, \text{Sm}_{ucd}$  or  $\text{Sch}_{ucd}$ , then there is a representable hypercover  $K_\bullet$  in the same site and a morphism  $L_\bullet \rightarrow K_\bullet$  so that  $\mathbb{Z}L^\bullet$  is quasi-isomorphic to  $\mathbb{Z}K^\bullet$ .*

*Proof.* For a finite family  $\{U_i\}$ ,

$$\coprod \text{Hom}(\_, U_i) \rightarrow \text{Hom}(\_, \coprod U_i)$$

is a Zariski cover, so in particular it is an  $h$ - and  $ucd$ -cover. Thus they have the same associated sheaves of abelian groups.

In addition, if

$$\coprod \text{Hom}(\_, U_i) \rightarrow \coprod \text{Hom}(\_, V_j)$$

is a morphism of finitely representable presheaves, then Yoneda's lemma tells us the identity morphisms  $\text{id}_{U_i} \in \text{Hom}(U_i, U_i)$  determine the diagonal in the commutative diagram

$$\begin{array}{ccc} \coprod \text{Hom}(\_, U_i) & \longrightarrow & \coprod \text{Hom}(\_, V_j) \\ \downarrow & \nearrow & \downarrow \\ \text{Hom}(\_, \coprod U_i) & \dashrightarrow & \text{Hom}(\_, \coprod V_j) \end{array}$$

and thus the dashed arrow. Hence every morphism of finitely representable presheaves determines a morphism of associated representable coproducts (but not vice versa!) and these morphisms are the same on passing to associated sheaves.

Thus given a finitely representable hypercover  $L_\bullet$  with  $L_n = \coprod \text{Hom}(\_, U_{n,i})$ , take  $K_\bullet$  with  $K_n = \text{Hom}(\_, \coprod U_{n,i})$  with simplicial morphisms given as above. It is representable and yields the same complex of sheaves of abelian groups (it in fact is a local acyclic fibration in the Zariski topology, since it locally has sections.)  $\square$

**Corollary 3.18.** *Let  $X_\bullet$  be a representable simplicial presheaf in  $\text{Sch}_h$ . Let  $R_{\text{Sm}}(X_\bullet)$  be the subcategory of  $FR_{\text{Sm}}(X_\bullet) = \{\text{finitely representable local acyclic fibrations in } \text{Sm}_h\}$  whose components are representable. Then every  $L_\bullet \in FR_{\text{Sm}}(X_\bullet)$  has a quasi-isomorphism  $\mathbb{Z}L^\bullet \rightarrow \mathbb{Z}K^\bullet$  for some  $K_\bullet \in R_{\text{Sm}}(X_\bullet)$ , so for a bounded below complex of sheaves of abelian groups  $\mathcal{F}^\bullet$  with the filtration bête*

$$\varinjlim_{K_\bullet \in hR_{\text{Sm}}(X_\bullet)} H^p(\text{Tot Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet)) \simeq \text{Ext}^p(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet)$$

and there is a filtered quasi-isomorphism of ind-objects in the derived category

$$\varinjlim_{K_\bullet \in hR_{\text{Sm}}(X_\bullet)} \text{Tot Hom}(\mathbb{Z}K^\bullet, \mathcal{F}^\bullet) \simeq R\text{Hom}(\mathbb{Z}X^\bullet, \mathcal{F}^\bullet).$$

*Proof.* Proposition 2.19 says we may compute using finitely representable hypercovers in  $\mathrm{Sm}_h$ . The lemma says finitely representable hypercovers have associated complexes of sheaves of free abelian groups equivalent to those of representable hypercovers.  $\square$

#### 4. ALGEBRAIC DE RHAM COMPLEX

4.1.  $\Omega^q$  is an  $h$ -sheaf. For every  $q \geq 0$ , let  $\Omega^q$  denote the presheaf on the site  $\mathrm{Sm}_h$  given by

$$X \mapsto \Gamma(X, \Omega_{X/\mathbb{C}}^q).$$

It is a presheaf of  $\mathcal{O}$ -modules.

**Lemma 4.1.** *If  $f : X \rightarrow Y$  is a dominant morphism of smooth  $\mathbb{C}$ -schemes, then  $\Omega^q(Y) \hookrightarrow \Omega^q(X)$ .*

*Proof.* Suppose  $\omega \in \Omega^q(Y)$  has  $f^*\omega = 0$ . Generic smoothness gives a Zariski open dense  $U \subset Y, V = f^{-1}(U) \subset X$  where  $f|_V$  is smooth. Then  $f|_V^*$  is injective so we see  $\omega$  vanishes on an open dense set, so must be zero.  $\square$

**Proposition 4.2.**  *$\Omega^q$  is a sheaf in  $\mathrm{Sm}_h$ .*

*Proof.* We must check, for every covering sieve  $R$  of  $X$ , that  $\Omega^q(R) = \Omega^q(X)$ . We may assume  $X$  is irreducible. It is enough to check for  $R$  generated by a single  $h$ -covering family, and in fact a single covering  $u : Y \rightarrow X$ : if  $\{U_i \rightarrow X\}$  is a finite covering family, then  $\Omega^q(\{U_i \rightarrow X\}) = \Omega^q(\coprod_j U_j)$  because  $\Omega^q$  is already a Zariski sheaf, and  $U_i \rightarrow \coprod_j U_j$  is a Zariski covering. Since every  $f \in R$  factors through  $u$ , the  $R$ -local sections are just elements  $\omega \in \Omega^q(Y)$  which, for every pair of maps  $f, g : Z \rightrightarrows Y$  with  $uf = ug$ , we have  $f^*\omega = g^*\omega$ .

We first check the case where  $u$  is a smooth morphism. In this case all pairs  $f, g$  factor through the smooth  $W = Y \times_X Y \rightrightarrows Y$ , so it is enough to check for  $Z = W$ . For  $q = 0$ , this is the usual exact sequence of algebras

$$0 \rightarrow A \rightarrow B \rightarrow B \otimes_A B$$

where  $A \hookrightarrow B$  is the injective map coming from a dominant morphism. For  $q = 1$  we have from the usual exact sequences of differentials the diagram

$$\begin{array}{ccccccc}
 0 & & & & & & \\
 & \searrow & & & & & \\
 0 & \longrightarrow & \Omega^1(X) & \longrightarrow & \Omega^1(Z) & \longrightarrow & \Gamma(Z, \Omega_{Z/X}^1) = p_1^* \Gamma(Y, \Omega_{Y/X}^1) \oplus p_2^* \Gamma(Y, \Omega_{Y/X}^1) \\
 & & \searrow & & \uparrow \uparrow & \nearrow & \\
 & & & & \Omega^1(Y) & & 
 \end{array}$$

Thus  $\Omega^1(X) \hookrightarrow \Omega^1(Y)$ , and clearly the image is contained in the equalizer of the two vertical arrows. Conversely, if a form  $\omega \in \Omega^1(Y)$  is sent by both vertical arrows to  $\eta \in \Omega^1(Z)$ , then commutativity of the right triangle gives that  $\omega$  must be sent to the same place by the pair of diagonal arrows. But the only thing in the intersection of the image of  $p_1^*$  and  $p_2^*$  is zero, hence  $\eta$  must lift to a form in  $\Omega^1(X)$ , so  $\Omega^1(X)$  is precisely the equalizer of the vertical arrows. The cases  $q > 0$  follow from applying the (exact) wedge product functor.



For general  $u$ , the lemma gives  $\Omega^1(X) \hookrightarrow \Omega^1(Y)$ . The image of  $\Omega^1(X)$  is by definition in the intersection of all equalizers. Conversely, suppose  $\omega \in \Omega^1(Y)$  is in the equalizer of every pair of arrows  $f, g : Z \rightrightarrows Y \rightarrow X$ . Generic smoothness and the case of a smooth morphism show that the result is true at the generic point. The proposition then follows from the following lemma.  $\square$

**Lemma 4.3.** *Suppose  $f : Y \rightarrow X$  is an  $h$ -covering of smooth  $\mathbb{C}$ -schemes with  $X$  irreducible. Let  $\{Y_i\}$  be the set of components of  $Y$  which dominate  $X$ . Then the diagram*

$$\begin{array}{ccc} \Gamma(X, \Omega_X^q) & \hookrightarrow & \bigoplus_i \Gamma(Y_i, \Omega_{Y_i}^q) \\ \downarrow & & \downarrow \\ \Omega_X^q \otimes_{\mathbb{C}} k(X) & \hookrightarrow & \bigoplus_i \Omega_{Y_i}^q \otimes_{\mathbb{C}} k(Y_i) \end{array}$$

*is cartesian: if a  $q$ -form  $\omega$  on the generic point of  $X$  lifts to a  $q$ -form on the generic point of  $Y$  that extends to all of  $Y$ , then  $\omega$  extends to all of  $X$ .*

*Proof.* By Hartog's theorem<sup>2</sup> we may safely throw out codimension  $\geq 2$  subsets of  $X$ . Hence if  $X' \subset X$  is the open set where  $\omega \in \Omega_X^q \otimes_{\mathbb{C}} k(X)$  is defined, we may assume the complement  $D = X - X'$  is a union of finitely many smooth divisors (throwing out singular and intersection sets.) We may extend over one divisor at a time, so assume  $D$  is a single smooth divisor.

Note it is enough to prove the lemma after replacing  $Y$  with any subscheme which dominates  $X$  so that  $E = f^{-1}(D)$  is non-empty. Throwing out closed subsets we may assume  $E$  is a divisor. Let  $\phi : E \rightarrow D$  be  $f$  restricted to  $E$ . Generic smoothness gives a point  $y \in E$  where  $\phi$  is smooth over  $x = \phi(y) \in D$ . We choose a complementary subspace to  $m_{E,y}/m_{E,y}^2 \subset m_{Y,y}/m_{Y,y}^2$ , and lift generators of this subspace to equations  $g_1, \dots, g_r$  in  $\mathcal{O}_{Y,y}$ . We replace  $Y$  with a subvariety defined by the  $g_i$  in some neighborhood of  $y \in Y$  where the  $g_i$  are defined, so we can assume  $\dim Y = \dim X$ , and throwing out codim  $\geq 2$  points of  $X$  and closed subsets of  $Y$  we may assume that  $Y$  is smooth and connected,  $E$  is a smooth connected divisor, and  $\phi$  is étale at  $y$ .

The theorem on the dimension of fibers of a morphism ([15, II ex 3.22]) gives the subset of  $U \subset X$  where  $f$  is quasi-finite is open. The complement  $C = X - U$  is at worst dimension  $\dim X - 1$ . If it is equal to  $\dim X - 1$ , then its preimage is also  $\dim X - 1 = \dim Y - 1$ , so applying the theorem again to components of  $C$  we get a dense open set of  $C$  where  $f$  is quasi-finite: thus the subset of  $X$  where  $f$  is not quasi-finite is at least codimension 2 and we may safely throw that out, so we may assume  $f$  is quasi-finite.

By Zariski's Main Theorem ([9, 4.4.3] or [20, III.9.I]) we have a factorization  $Y \subset \text{Sh } X \xrightarrow{\pi} X$  where  $Y$  is an open immersion in the normalization  $\text{Sh } X$  of  $X$  in  $k(Y)$ . Let  $E' = \text{Sh } X - Y$ . Let  $W = \pi(E') - D$ . Since by Hartog's theorem we only have to extend across the generic point of  $D$ , we may throw out  $W$ . Hence we may assume  $\pi(E') \cap D$  is either empty or else is all of  $D$ . Throwing out more points we may assume  $E$  and  $E'$  are disjoint smooth divisors. Again we only have to extend over the generic point of  $D$ , so we may assume  $X$  and  $\text{Sh } X$  are affine. Let  $h, h'$  be defining equations for  $E, E'$ ; these exist since the stalk of  $f_* \mathcal{O}_Y$  over  $\mathcal{O}_{X,D}$  is a semi-local PID. We may assume  $h|_{E'} = 1 = h'|_E$ .

<sup>2</sup>Or the algebraic version regarding normal varieties and codimension  $\geq 2$  sets, see [15, II.8.19].

We have an  $\omega \in \Omega_{X/\mathbb{C}}^q \otimes_{\mathbb{C}} k(X)$  so that  $f^*\omega$  extends to an  $\eta \in \Gamma(Y, \Omega_{Y/\mathbb{C}}^q)$ . Then for some  $m$  large enough  $h'^m \eta \in \Gamma(\text{Sh } X, \Omega_{\text{Sh } X/\mathbb{C}}^q)$ . The theory of traces of  $q$ -forms (for example [19, 4.6.7]) gives us a  $q$ -form on  $X$

$$\text{trace}(h'^m \eta).$$

Away from  $D$ , we have

$$\text{trace}(h'^m \eta) = \text{trace}(h'^m) \omega$$

so it is enough to show that  $\text{trace}(h'^m)$  is invertible. Since we can throw out closed subsets not containing  $D$ , it is enough to show  $\text{trace}(h'^m)|_D$  is invertible. But this is just

$$e_{E/D} \text{trace}_{E/D}(h'|_E)^m = e_{E/D} \deg(E \rightarrow D)^m$$

since  $h'|_E = 1$ , where  $e_{E/D}$  is the ramification.  $\square$

*Remark 4.4.* We have a complex of sheaves  $\Omega^\bullet$  on  $\text{Sm}_h$  and an augmentation

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{O} \rightarrow \Omega^1 \rightarrow \Omega^2 \rightarrow \dots$$

coming from the usual inclusions and exterior differentiation. The complex  $\Omega^\bullet$  has a natural filtration, the filtration bête.

Fix an  $X \in \text{Sch}$ . For simplicity we assume  $X$  is irreducible. We consider the sites  $\text{Sm}_h/X$  of smooth  $\mathbb{C}$ -schemes over  $X$ ,  $\text{Sch}_h/X$  all  $\mathbb{C}$ -schemes over  $X$ , and  $X_{\text{Zar}}$  the small site of Zariski-open subsets of  $X$ . The natural inclusion  $\gamma : X_{\text{Zar}} \hookrightarrow \text{Sch}_h/X$  gives  $X_{\text{Zar}}$  the induced Grothendieck topology, since a family of Zariski open sets is a Zariski cover only if it is an  $h$ -cover. Therefore  $\gamma$  is continuous [11, ex III 3.1] and induces a geometric morphism of topoi [11, ex III 1.2.1] which we also denote by  $\gamma$ :

$$\gamma = (\gamma^*, \gamma_*) : \text{Sh } \text{Sm}_h/X \simeq \text{Sh } \text{Sch}_h/X \rightarrow \text{Sh } X_{\text{Zar}}$$

the first equivalence being given by Corollary 3.7. Perhaps confusingly, for an  $h$ -sheaf  $F$  we have  $\gamma_* F = F \circ \gamma$ . Note that

$$\mathbb{Z}X_h = \gamma^* \mathbb{Z}X_{\text{Zar}}$$

as both are the sheaf of free abelian groups associated to the constant presheaf with value  $\mathbb{Z}$ .

*Remark 4.5.* Since  $\Omega^q$  is a sheaf on  $\text{Sm}_h$ , for any  $X \in \text{Sch}$  and any diagram

$$X \leftarrow X_0 \rightrightarrows X_0 \times_X X_0 \leftarrow X_1$$

where  $X_0 \rightarrow X$  is an  $h$ -covering and  $X_0, X_1 \in \text{Sm}$ ,  $\gamma_* \Omega_X^q$  is determined by the exact sequence

$$0 \rightarrow \Gamma(X, \gamma_* \Omega_X^q) \rightarrow \Gamma(X_0, \Omega_{X_0/\mathbb{C}}^q) \rightrightarrows \Gamma(X_1, \Omega_{X_1/\mathbb{C}}^q).$$

This shows  $\gamma_* \Omega_X^q$  is quasi-coherent. Since by [16, 2] we can choose proper  $h$ -covers,  $\gamma_* \Omega_X^q$  is coherent.

#### 4.2. Results of Du Bois.

**Definition 4.6.** Let  $X$  be a  $\mathbb{C}$ -scheme. A *good cover* of  $X$  is a smooth representable  $h$ -hypercover  $Z_\bullet \rightarrow X$  with components quasi-projective and proper over  $X$ .

**Theorem 4.7** ([4, 3.11]). *Let  $X$  be a  $\mathbb{C}$ -scheme, and  $e : K_\bullet \rightarrow X, e' : K'_\bullet \rightarrow X$  two good covers of  $X$ . Let  $\alpha : K'_\bullet \rightarrow K_\bullet$  be a map over  $X$ . Then the induced map*

$$Re_*(\Omega_{K_\bullet/\mathbb{C}}^p) \rightarrow Re'_*(\Omega_{K'_\bullet/\mathbb{C}}^p)$$

*is an isomorphism in the derived category.*

The morphism is constructed by applying  $Re_*$  to

$$\Omega_{K_\bullet/\mathbb{C}}^p \rightarrow \alpha_* \Omega_{K'_\bullet/\mathbb{C}}^p \rightarrow R\alpha_* \Omega_{K'_\bullet/\mathbb{C}}^p.$$

This direct image is computed in the Zariski topology; by GAGA [14] this commutes with analytification, since all components are proper over the base  $X$ .

**Corollary 4.8** ([4, 3.17]). *Same hypotheses as above. Giving the complexes  $\Omega_{K_\bullet/\mathbb{C}}, \Omega_{K'_\bullet/\mathbb{C}}$  the filtration *bête*, the canonical map*

$$Re_*(\Omega_{K_\bullet/\mathbb{C}}) \rightarrow Re'_*(\Omega_{K'_\bullet/\mathbb{C}})$$

*is an isomorphism in the filtered derived category.*

For  $X$  smooth, we can take  $K_\bullet = X_\bullet$  the constant simplicial scheme; this clearly is a smooth resolution of  $X$ . In this case the theorems degenerate to

**Proposition 4.9.** *For  $X$  a smooth  $\mathbb{C}$ -scheme and good cover  $e' : K'_\bullet \rightarrow X$ , we have*

$$e'_* \Omega_{K'_\bullet/\mathbb{C}}^q = \Omega_{X/\mathbb{C}}^q$$

*and  $R^i e'_* \Omega_{K'_\bullet/\mathbb{C}}^q = 0$  for  $i > 0$ .*

**Corollary 4.10.** *Same hypotheses as above. Giving the complexes  $\Omega_{K'_\bullet/\mathbb{C}}, \Omega_{X/\mathbb{C}}$  the filtration *bête*, the canonical map*

$$Re_*(\Omega_{K'_\bullet/\mathbb{C}}) \rightarrow \Omega_{X/\mathbb{C}}$$

*is an isomorphism in the filtered derived category.*

**4.3. Comparison of  $h$ - and Zariski topology.** The following result comes from generous suggestion of Alexander Beilinson.

**Theorem 4.11.** *Let  $X$  be a  $\mathbb{C}$ -scheme, and  $\mathcal{F}^\bullet$  a bounded below complex of sheaves of abelian groups in  $\mathrm{Sm}_h/X$  given the filtration *bête*. Let  $Q(X)$  be the subcategory of good covers of  $X$  in  $R\mathrm{Sm}(X)$  the category of representable smooth  $h$ -hypercovers of  $X$ . Then the associated homotopy category of cochain complexes  $hQ(X)$  is cofiltered, and there is a filtered quasi-isomorphism of ind-objects*

$$"\varinjlim_{Z_\bullet \in hQ(X)} R\mathrm{Hom}_{\mathrm{Zar}}(\mathbb{Z}Z^\bullet, \mathcal{F}^\bullet)) \simeq R\mathrm{Hom}_h(\mathbb{Z}X, \mathcal{F}^\bullet).$$

*Proof.* We construct, for any smooth representable  $h$ -hypercover  $K_\bullet \rightarrow X$ , a diagram

$$Z_\bullet \xleftarrow{\pi} L_\bullet \xrightarrow{\phi} K_\bullet$$

where  $L_\bullet$  is a smooth representable  $h$ -hypercover of  $X$ ,  $Z_\bullet$  is a good cover of  $X$ , and  $\pi$  is a local acyclic fibration in the Zariski topology. Assuming such a construction exists, then the  $L_\bullet$  are cofinal in all smooth representable hypercovers, so by Corollary 3.18 we have a filtered quasi-isomorphism

$${}^{\text{''}}\varinjlim_{L_\bullet} \text{Tot Hom}(\mathbb{Z}L^\bullet, \mathcal{F}^\bullet) \simeq R\text{Hom}_h(\mathbb{Z}X, \mathcal{F}^\bullet).$$

Note of course  $\text{Tot Hom}(\mathbb{Z}L^\bullet, \_)$  does not see the topology.

Now there is also a natural morphism

$${}^{\text{''}}\varinjlim_{L_\bullet} \text{Tot Hom}(\mathbb{Z}L^\bullet, \mathcal{F}^\bullet) \rightarrow {}^{\text{''}}\varinjlim_{Z_\bullet} R\text{Hom}_{\text{Zar}}(\mathbb{Z}Z^\bullet, \mathcal{F}^\bullet)$$

where the  $Z_\bullet$  run over good covers of  $X$ . Each  $L_\bullet$  is a Zariski local acyclic fibration of some  $Z_\bullet$ , which gives the map. We claim in the limit this is a filtered quasi-isomorphism. This is because we can compute  $R\text{Hom}_{\text{Zar}}(\mathbb{Z}Z^\bullet, \mathcal{F}^\bullet)$  as the limit of Čech cohomology over Zariski local fibrations  $L_\bullet \rightarrow Z_\bullet$  by Corollary 3.18, and every such  $L_\bullet$  appears on the left side. The composition gives the desired filtered quasi-isomorphism

$${}^{\text{''}}\varinjlim_{Z_\bullet \in hQ(X)} R\text{Hom}_{\text{Zar}}(\mathbb{Z}Z^\bullet, \mathcal{F}^\bullet) \simeq R\text{Hom}_h(\mathbb{Z}X, \mathcal{F}^\bullet).$$

To construct the  $L_\bullet$  and the  $Z_\bullet$ , in degree zero we form the diagram of smooth  $\mathbb{C}$ -schemes

$$\begin{array}{ccccc} Z_0 & \xleftarrow{\text{Zar}} & L_0 & \longrightarrow & K_0 \\ & \searrow \text{proper} & & \swarrow h\text{-cover} & \\ & & X & & \end{array}$$

which exists by Corollary 3.15. (Here “Zar” indicates a Zariski open cover and “proper” indicates a proper surjective cover.) Assume inductively we have constructed a diagram of  $n$ -truncated objects

$$Z_{\leq n} \xleftarrow{\pi_{\leq n}} L_{\leq n} \rightarrow K_{\leq n}$$

where

- (1) All objects are smooth representable and the  $K_{\leq n}$  is the truncation of the  $K_\bullet$ .
- (2)  $Z_{\leq n}$  is an  $n$ -truncated good cover of  $X$ .
- (3)  $L_i \rightarrow Z_i$  is a Zariski open cover for all  $0 \leq i \leq n$ .

Note that 3 implies  $L_{\leq n} \rightarrow Z_{\leq n}$  is a local acyclic fibration: for  $k > n$  the condition

$$\begin{array}{ccc} \partial\Delta^k & \longrightarrow & L_{\leq n}(U) \\ \downarrow & \nearrow & \downarrow \\ \Delta^k & \longrightarrow & Z_{\leq n}(U) \end{array}$$

is empty, and for  $k \leq n$  a local section  $Z_k(V) \rightarrow L_k(V)$  allows us to lift.

We claim that these assumptions imply  $(i_{n*}L_{\leq n})_{n+1} \rightarrow (i_{n*}Z_{\leq n})_{n+1}$  is an open Zariski cover. This comes from the following fact: if we have a commutative diagram

$$\begin{array}{ccccc}
 & & B' & & \\
 & & \downarrow & \searrow & \\
 & & & B & \\
 A' & \xrightarrow{\quad} & C' & \searrow & \\
 & \searrow & & \downarrow & \\
 & A & \xrightarrow{\quad} & C & 
 \end{array}$$

where all the diagonal arrows are Zariski covers, then  $A' \times_{C'} B' \rightarrow A \times_C B$  is a Zariski cover. To see this, first we get map  $A' \rightarrow A \times_C C'$ . This is a Zariski open cover because the composite with the projection to  $A$  is an open cover, and  $A \times_C C' \rightarrow A$  itself is a Zariski open cover.<sup>3</sup> Likewise for  $B' \rightarrow B \times_C C'$ . Thus the map

$$A' \times_{C'} B' \rightarrow (A \times_C C') \times_{C'} (B \times_C C') = (A \times_C B) \times_C C'$$

is a Zariski open cover. But  $(A \times_C B) \times_C C' \rightarrow A \times_C B$  is a Zariski open cover by base change, hence so is  $A' \times_{C'} B' \rightarrow A \times_C B$ .

Now the  $i_{n*}$  are constructed by finite products and fiber products of the  $L_k$ 's and  $Z_k$ 's, and the morphism is component-by-component, and these are all Zariski open covers. Hence repeating the argument above will show that  $(i_{n*}L_{\leq n})_{n+1} \rightarrow (i_{n*}Z_{\leq n})_{n+1}$  is a Zariski cover.

Now construct the diagram

$$\begin{array}{ccccc}
 & & C & \xrightarrow{\quad} & B \\
 & \swarrow \text{Zar} & & \swarrow \text{Zar} & \searrow \\
 & Z'_{n+1} & & L'_{n+1} & K_{n+1} \\
 \text{proper} \downarrow & & & \downarrow h & \downarrow h \\
 (i_{n*}Z_{\leq n})_{n+1} & \xleftarrow{\quad \text{Zar} \quad} & (i_{n*}L_{\leq n})_{n+1} & \xrightarrow{\quad} & (\text{cosk}_n K)_{n+1}
 \end{array}$$

where

- (1) Zar indicates an arrow is a Zariski open cover, “proper” a proper surjective cover, and  $h$  an  $h$ -cover;
- (2)  $B$  is the fiber product  $K_{n+1} \times_{(\text{cosk}_n K)_{n+1}} (i_{n*}L_{\leq n})_{n+1}$ ;
- (3)  $C \xrightarrow{\text{Zar}} Z'_{n+1} \xrightarrow{\text{proper}} (i_{n*}Z_{\leq n})_{n+1}$  is the factorization of the  $h$ -covering  $B \rightarrow (i_{n*}Z_{\leq n})_{n+1}$  given by Corollary 3.15, so  $Z'_{n+1}$  is smooth representable proper over  $(i_{n*}Z_{\leq n})_{n+1}$  with quasi-projective components;
- (4)  $L'_{n+1}$  is the fiber product  $C \times_{(i_{n*}Z_{\leq n})_{n+1}} (i_{n*}L_{\leq n})_{n+1}$ . In particular, it is an open Zariski cover of  $C$ , and hence smooth.

Then up to degeneracies, the  $L'_{n+1}$  and  $Z'_{n+1}$  satisfy all the conditions needed: the  $Z'_{n+1}$  is quasi-projective and proper and surjective over the coskeleton and  $X$  and

<sup>3</sup>It is an étale surjective monomorphism on components.

$L'_{n+1}$  is a Zariski cover of  $Z'_{n+1}$  and completes  $L_{\leq n}$  to a truncated  $h$ -hypercover. By Remark 2.16 we can fulfill degeneracy conditions by adding disjoint unions with lower degree pieces, which does not affect any of the properties we have established.

Finally we have to show the map  $L'_{n+1} \rightarrow K_{n+1}$  is compatible with the face maps, in other words, the direct map  $L'_{n+1} \rightarrow (i_{n*}L_{\leq n})_{n+1}$  given by the vertical arrow factors through  $B$ , e.g. is equal to

$$L'_{n+1} \rightarrow C \rightarrow B \rightarrow (i_{n*}L_{\leq n})_{n+1}.$$

This can be checked as follows: first we simplify the notation in the diagram

$$\begin{array}{ccc} & C & \xrightarrow{\quad} B \\ & \swarrow & \searrow \\ Z & & L \\ \downarrow & & \downarrow \\ \overline{Z} & \xleftarrow{\text{Zar}} & \overline{L} \end{array}$$

where  $Z = Z_{n+1}$ ,  $\overline{Z} = (i_{n*}Z_{\leq n})_{n+1}$  and likewise for  $L$ . We write composites  $C \rightarrow B \rightarrow \overline{L}$  as  $C\overline{B}\overline{L}$ , et cetera. Then by construction  $C\overline{B}\overline{L}\overline{Z} = CZ\overline{Z}$ , and  $LCZ\overline{Z} = L\overline{L}\overline{Z}$ . Hence  $LC\overline{B}\overline{L}\overline{Z} = L\overline{L}\overline{Z}$ ; but  $\overline{L}\overline{Z}$  is an epimorphism, so we can right-cancel, yielding  $LC\overline{B}\overline{L} = L\overline{L}$ , which is what we wanted. Hence the map  $L'_{n+1} \rightarrow K_{n+1}$  is consistent with face maps. The degeneracies are automatic by the splitting construction. Thus completes the construction and the proof.  $\square$

#### 4.4. Algebraic de Rham complex.

**Proposition 4.12.** *Let  $X$  be a  $\mathbb{C}$ -scheme. Then for any good cover  $Z_{\bullet}$  of  $X$ , we have a filtered quasi-isomorphism*

$$R\mathrm{Hom}_{\mathrm{Zar}}(\mathbb{Z}Z^{\bullet}, \Omega^{\bullet}) \simeq R\mathrm{Hom}_h(\mathbb{Z}X, \Omega^{\bullet}).$$

*Proof.* By the result of Du Bois (Corollary 4.8), every term in the ind-object of Theorem 4.11

$$\text{"}\varinjlim\text{"}_{Z_{\bullet} \in hQ(X)} R\mathrm{Hom}_{\mathrm{Zar}}(\mathbb{Z}Z^{\bullet}, \Omega^{\bullet})$$

is isomorphic (recall  $Q(X)$  is the category of Du Bois covers of  $X$ .)  $\square$

**Definition 4.13.** Let  $e : Z_{\bullet} \rightarrow X$  be a good cover of  $X$ . Define the algebraic de Rham complex as

$$\underline{\Omega}_X := Re_*\Omega_{K_{\bullet}/\mathbb{C}}.$$

Recall  $\gamma_* : \mathrm{Sh} \mathrm{Sm}_h/X \rightarrow \mathrm{Sh} X_{\mathrm{Zar}}$  is the direct image of sheaves on the smooth  $h$ -site over  $X$  to sheaves on the small Zariski site of  $X$ .

**Theorem 4.14.** *The algebraic de Rham complex is quasi-isomorphic in the filtered derived category to  $R\gamma_*\Omega$  with the filtration bête.*

*Proof.* Apply the previous proposition Zariski locally on the base  $X$ .  $\square$

**Corollary 4.15.** *Algebraic de Rham cohomology, with the filtration bête, is computed by the hypercohomology of  $\Omega$  in  $\mathrm{Sm}_h/X$  with the filtration bête.*

*Proof.*

$$\mathbf{H}_h^i(\mathrm{Sm}_h/X, \Omega^\cdot) = \mathbf{H}^i(X_{\mathrm{Zar}}, R\gamma_*\Omega^\cdot) = \mathbf{H}^i(X_{\mathrm{Zar}}, Re_*\Omega_{K_\bullet/\mathbb{C}}^\cdot) = H_{dR}^i(X).$$

□

*Remark 4.16.* As noted before, this filtration is typically not the Hodge filtration.

#### 4.5. Questions.

- (1) For an open  $\mathbb{C}$ -scheme  $X$ , is there a site of “log  $h$ -covers of  $X$ ” which takes the place of Deligne’s construction of smooth hypercovers with boundary a normal crossing divisor?
- (2) Is there a model-theoretic generalization of Lemma 2.7?
- (3) What are the minimum hypotheses about  $\Omega^q$  which allow the Du Bois results to go through? Is the following enough:  $\mathcal{F}$  is sheaf of  $\mathcal{O}$ -modules on  $\mathrm{Sm}_h$ , locally free on smooth Zariski sites, with “transfers?”
- (4) Is there a difference between the  $\mathrm{Sm}_{ucd}$  and  $\mathrm{Sm}_h$ ?
- (5) Is there a characterization of hypercovers in terms of ordinary covers if one works with the geometric realization?
- (6) The genesis of all of this work was an idea of Nori, on “holomorphic Whitney forms.” The basic idea was to look at functionals on cycles which “vary holomorphically,” in analogy with [23]; a discussion will be forthcoming in a future article. What is the relationship between this theory, “holomorphic Whitney forms,” and intersection cohomology sheaves?

#### REFERENCES

- [1] Brian Conrad. Cohomological descent. Available at <http://www.math.lsa.umich.edu/~bdconrad/papers/hypercover.pdf>.
- [2] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, (83):51–93, 1996.
- [3] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, 44:5–77, 1974.
- [4] Philippe Du Bois. Complexe de de Rham filtré d’une variété singulière. *Bull. Soc. Math. France*, 109(1):41–81, 1981.
- [5] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen. Hypercovers and simplicial presheaves. *Math. Proc. Cambridge Philos. Soc.*, 136(1):9–51, 2004.
- [6] Daniel Dugger and Daniel C. Isaksen. Weak equivalences of simplicial presheaves. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 97–113. Amer. Math. Soc., Providence, RI, 2004.
- [7] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.
- [8] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.
- [9] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.
- [10] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, 29:95–103, 1966.
- [11] A. Grothendieck, editor. *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269.
- [12] A. Grothendieck, editor. *Théorie des topos et cohomologie étale des schémas. Tome 2*. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 270.

- [13] A. Grothendieck, editor. *Théorie des topos et cohomologie étale des schémas. Tome 3.* Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 305.
- [14] A. Grothendieck, editor. *Revêtements étales et groupe fondamental (SGA 1).* Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960–61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].
- [15] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [16] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)* 79 (1964), 109–203; *ibid.* (2), 79:205–326, 1964.
- [17] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.
- [18] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1994. With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original.
- [19] Joseph Lipman. *Residues and traces of differential forms via Hochschild homology*, volume 61 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1987.
- [20] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.
- [21] Andrei Suslin and Vladimir Voevodsky. Relative cycles and Chow sheaves. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 10–86. Princeton Univ. Press, Princeton, NJ, 2000.
- [22] V. Voevodsky. Homology of schemes. *Selecta Math. (N.S.)*, 2(1):111–153, 1996.
- [23] Hassler Whitney. *Geometric integration theory.* Princeton University Press, Princeton, N. J., 1957.